A State Private University (Established under Act No. 41 of 2013)
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Department of Electrical and Electronics Engineering (EEE)

Lecture Notes

Control System Engineering (EEE2510)

AY 2026-27: V Sem, III Year EEE

MODULE 1: INTRODUCTION & MODELING

Mathematical models of physical systems, transfer function derivation, block diagram reduction techniques, and signal flow graph analysis.

Dr. Saravanakumar Gurusamy
Associate Professor of EEE
saravanakumar@presidencyuniversity.in

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Module 1

System Components and their representation

Module 1: System Components and their representation

- Introduction to control systems
- Mathematical models of physical systems-differential equations of physical systems, Mechanical systems, Electrical systems,
- Block diagrams and signal flow graphs

Table 1.1: Course Outcomes and Bloom's Taxonomy Levels

CO	Course Outcome Statement	Bloom's Level
CO1	**Derive transfer functions for basic electrical and mechanical systems using standard modeling techniques**.	**Apply**
CO2	Solve control systems using block diagram reduction and Mason's Gain Formula for signal flow graphs.	Apply
CO3	Analyze the time response characteristics of second-order systems for standard test inputs.	Analyse
CO4	Analyze system stability using time-domain and frequency-domain methods.	Analyse
CO5	Select appropriate compensation techniques (lead, lag, lead-lag) to improve system performance.	Apply
CO6	Determine controllability and observability of state-space models using standard criteria.	Apply

§ 1.1 Laplace Transform - Review

The unilateral Laplace transform is the mathematical tool used to convert continuous-time differential equations into algebraic relationships in the s-domain.

The Laplace transform of a time function $f(t)$ is defined as:

$$\mathcal{L}\{f(t)\} = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt \tag{1.1}$$

where $s = \sigma + j\omega$ is a complex frequency variable. The inverse Laplace transform is defined as:

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds \tag{1.2}$$

To avoid performing complex contour integration, engineers rely on Laplace transform tables and theorems.

1.1.1 Laplace Transforms of Common Functions

Table 1.2 details the Laplace transforms of the most common physical signals.

Table 1.2: Laplace Transforms of Common Time Functions

Item No.	Time Function, $f(t)$	Laplace Transform, $F(s)$	Domain of Convergence
1	$\delta(t)$ (unit impulse)	1	All s
2	$u(t)$ (unit step)	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
3	$t\,u(t)$ (unit ramp)	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$
4	$t^n\,u(t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}(s) > 0$
5	$e^{-at}\,u(t)$ (exponential decay)	$\frac{1}{s+a}$	$\mathrm{Re}(s) > -a$
6	$\sin(\omega t)\,u(t)$ (sine wave)	$\frac{\omega}{s^2+\omega^2}$	$\mathrm{Re}(s) > 0$
7	$\cos(\omega t)\,u(t)$ (cosine wave)	$\frac{s}{s^2+\omega^2}$	$\mathrm{Re}(s) > 0$

1.1.2 Essential Laplace Transform Theorems

Table 1.3 summarizes the operational theorems used to manipulate frequency-domain expressions.

Table 1.3: Essential Laplace Transform Theorems

No.	Theorem Name	Time Domain, $f(t)$	Frequency Domain, $F(s)$
1	Linearity	$k_1 f_1(t) + k_2 f_2(t)$	$k_1 F_1(s) + k_2 F_2(s)$
2	Frequency Shift	$e^{-at}f(t)$	$F(s+a)$
3	Time Shift	$f(t-T)\,u(t-T)$	$e^{-sT}F(s)$
4	Scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
5	First Derivative	$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$
6	Second Derivative	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0^-) - f'(0^-)$
7	Integration	$\int_{0^-}^{t} f(\tau)d\tau$	$\frac{F(s)}{s}$
8	Initial Value Theorem	$f(0^+) = \lim_{t\to 0^+} f(t)$	$\lim_{s\to\infty} sF(s)$
9	Final Value Theorem	$f(\infty) = \lim_{t\to\infty} f(t)$	$\lim_{s\to 0} sF(s)$

§ 1.2 Partial-Fraction Expansion Techniques

To find the inverse Laplace transform of a complex rational function $F(s) = \frac{N(s)}{D(s)}$ where the order of $N(s)$ is strictly less than the order of $D(s)$, we expand it into a sum of simpler terms. If the order of $N(s)$ is greater than or equal to $D(s)$, long division must be performed first.

1.2.1 Case 1: Roots of the Denominator are Real and Distinct

If $D(s)$ has n real and distinct roots, we write:

$$F(s) = \frac{N(s)}{(s+p_1)(s+p_2)\dots(s+p_n)} = \frac{K_1}{s+p_1} + \frac{K_2}{s+p_2} + \cdots + \frac{K_n}{s+p_n} \tag{1.3}$$

To evaluate any specific residue K_i, we multiply both sides by $(s+p_i)$ and let $s \longrightarrow -p_i$:

$$K_i = \lim_{s \to -p_i} (s+p_i)F(s) \tag{1.4}$$

1.2.2 Case 2: Roots of the Denominator are Real and Repeated

If $D(s)$ has a root $-p_1$ of multiplicity r, the expansion must include terms of all powers of $(s+p_1)$ from 1 up to r:

$$F(s) = \frac{K_1}{(s+p_1)^r} + \frac{K_2}{(s+p_1)^{r-1}} + \cdots + \frac{K_r}{s+p_1} + \text{other terms} \tag{1.5}$$

The residues K_i associated with the repeated root are found using:

$$K_i = \frac{1}{(i-1)!} \lim_{s \to -p_1} \frac{d^{i-1}}{ds^{i-1}} \left[(s+p_1)^r F(s)\right] \tag{1.6}$$

1.2.3 Case 3: Roots of the Denominator are Complex Conjugates

If $D(s)$ has a quadratic term with complex roots, $s^2 + as + b$, we can expand it as:

$$F(s) = \frac{K_1}{s+p_1} + \frac{K_2 s + K_3}{s^2 + as + b} \tag{1.7}$$

The constants K_2 and K_3 are evaluated by multiplying by the common denominator and balancing coefficients. Alternatively, the term can be expanded using complex residues:

$$F(s) = \frac{K_1}{s+p_1} + \frac{K_2}{s+\alpha+j\beta} + \frac{K_2^*}{s+\alpha-j\beta} \tag{1.8}$$

where K_2 and K_2^* are complex conjugates. Taking the inverse Laplace transform of the complex terms and combining them yields an exponentially damped sinusoid:

$$f(t) = Ce^{-\alpha t}\cos(\beta t - \phi) \tag{1.9}$$

where $\phi = \arctan\left(\frac{\text{Im}(K_2)}{\text{Re}(K_2)}\right)$.

Example 1.2.1 (First Order Differential Equation)

Find the transfer function represented by the first-order differential equation:

$$\frac{dc(t)}{dt} + 2c(t) = r(t) \tag{1.10}$$

where $c(t)$ represents the system output and $r(t)$ represents the system input.

Detailed Step-by-Step Derivation

Step 1: Understand the Definition of a Transfer Function

The transfer function, denoted as $G(s)$, represents the algebraic ratio of the Laplace transform of the output, $C(s)$, to the Laplace transform of the input, $R(s)$, under the condition that **all initial conditions are zero**:

$$G(s) = \frac{C(s)}{R(s)} \tag{1.11}$$

Step 2: Apply the Laplace Transform to Both Sides

Applying the unilateral Laplace transform to both sides of the differential equation gives:

$$\mathcal{L}\left\{\frac{dc(t)}{dt} + 2c(t)\right\} = \mathcal{L}\{r(t)\} \tag{1.12}$$

Using the **linearity property** of the Laplace transform, we can distribute the transform operator to each term:

$$\mathcal{L}\left\{\frac{dc(t)}{dt}\right\} + 2\mathcal{L}\{c(t)\} = \mathcal{L}\{r(t)\} \tag{1.13}$$

Step 3: Apply the Differentiation Theorem

According to the Laplace differentiation theorem, the derivative of a continuous-time function transforms as follows:

$$\mathcal{L}\left\{\frac{dc(t)}{dt}\right\} = sC(s) - c(0^-) \tag{1.14}$$

where $c(0^-)$ is the initial value of the output at $t = 0^-$.

Step 4: Assume Zero Initial Conditions

By definition of the transfer function, we set all initial conditions to zero ($c(0^-) = 0$). This simplifies the differential term's transform to:

$$\mathcal{L}\left\{\frac{dc(t)}{dt}\right\} = sC(s) \tag{1.15}$$

Substituting this back into the algebraic equation along with $C(s) = \mathcal{L}\{c(t)\}$ and $R(s) = \mathcal{L}\{r(t)\}$ yields:

$$sC(s) + 2C(s) = R(s) \tag{1.16}$$

Step 5: Factor Out the Output Transform

Factor the common output transform, $C(s)$, from the left-hand side of the algebraic equation:

$$(s + 2)C(s) = R(s) \tag{1.17}$$

Step 6: Formulate the Transfer Function

Solving for the ratio of the output transform over the input transform ($\frac{C(s)}{R(s)}$) gives the final closed-form transfer function:

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s + 2} \tag{1.18}$$

§ 1.3 Tutorial on Mathematical Modeling of Electrical and Mechanical System

1.3.1 Modeling of Electrical System

Example 1.3.1 (RLC Circuit)

Classical Differential Equation Method

Note:-

This classical approach operates first in the time domain, applying physical laws (Kirchhoff's Voltage Law) to construct differential equations before converting them to algebraic frequency-domain equations.

For the series *RLC* circuit shown in Figure 1.1, find the transfer function relating the capacitor voltage, $V_C(s)$, to the input voltage, $V(s)$. Let the input voltage be $v(t)$ and the output voltage across the capacitor be $v_C(t)$.

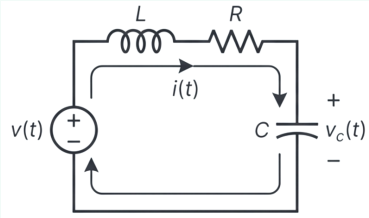


Figure 1.1: RLC Circuit.

Step-by-Step Derivation

Step 1: Write the KVL Loop Integro-Differential Equation

Summing the voltage drops across the inductor, resistor, and capacitor in the single-loop series network gives:

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t) \tag{1.19}$$

Step 2: Express Current in terms of Charge

To eliminate the integral and formulate a pure differential equation, we substitute the relationship between loop current $i(t)$ and charge $q(t)$, where $i(t) = \frac{dq(t)}{dt}$:

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = v(t) \tag{1.20}$$

Step 3: Relate Charge to the Capacitor Output Voltage

Since our designated output is the capacitor voltage $v_C(t)$, we use the capacitor charge-voltage relationship $q(t) = Cv_C(t)$ to change variables:

$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t) \tag{1.21}$$

Step 4: Take the Laplace Transform (Zero Initial Conditions)

Assuming all initial conditions are zero ($v_C(0^-) = 0$ and $\dot{v}_C(0^-) = 0$), we take the Laplace transform of the second-order differential equation:

$$LCs^2V_C(s) + RCsV_C(s) + V_C(s) = V(s) \tag{1.22}$$

Factoring out $V_C(s)$ on the left-hand side yields:

$$(LCs^2 + RCs + 1)V_C(s) = V(s) \tag{1.23}$$

Step 5: Solve for the Transfer Function

The transfer function $G(s)$ is the ratio of output to input:

$$G(s) = \frac{V_C(s)}{V(s)} = \frac{1}{LCs^2 + RCs + 1} \tag{1.24}$$

To express this in standard second-order form, we divide both the numerator and denominator by LC:

$$G(s) = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \tag{1.25}$$

Frequency-Domain Impedance Method

Note:-

This modern approach bypasses time-domain calculus completely by transforming the schematic directly into a Laplace-domain circuit and using equivalent electrical impedances.

Step-by-Step Derivation

Step 1: Determine the Component Impedances

We associate each passive circuit element with its Laplace impedance, $Z(s) = \frac{V(s)}{I(s)}$:

- **Resistor:** $Z_R(s) = R$

- **Inductor:** $Z_L(s) = Ls$
- **Capacitor:** $Z_C(s) = \frac{1}{Cs}$

Step 2: Write the Laplace Mesh Equation

Treating the complex impedances as general resistances in series, the mesh equation in the frequency domain is written by direct inspection:

$$(Z_L(s) + Z_R(s) + Z_C(s))\, I(s) = V(s) \tag{1.26}$$

Substituting the component impedances yields:

$$\left(Ls + R + \frac{1}{Cs}\right) I(s) = V(s) \tag{1.27}$$

Step 3: Relate Loop Current to the Output Voltage

The output voltage $V_C(s)$ is the voltage across the capacitor impedance, defined by Ohm's Law in the s-domain:

$$V_C(s) = I(s)Z_C(s) = I(s)\left(\frac{1}{Cs}\right) \implies I(s) = CsV_C(s) \tag{1.28}$$

Step 4: Substitute and Simplify

Substituting the expression for $I(s)$ into the mesh equation:

$$\left(Ls + R + \frac{1}{Cs}\right) CsV_C(s) = V(s) \tag{1.29}$$

Distributing the term Cs into the parentheses:

$$(LCs^2 + RCs + 1)V_C(s) = V(s) \tag{1.30}$$

Step 5: Formulate the Transfer Function

Solving for the ratio of $V_C(s)/V(s)$ gives the identical transfer function:

$$G(s) = \frac{V_C(s)}{V(s)} = \frac{1}{LCs^2 + RCs + 1} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \tag{1.31}$$

Multi-Loop Network via Mesh Analysis

Example 1.3.2 (Cascade RL-RC Circuit)

Consider the two-loop electrical network shown in Figure 1.2 and 1.3,

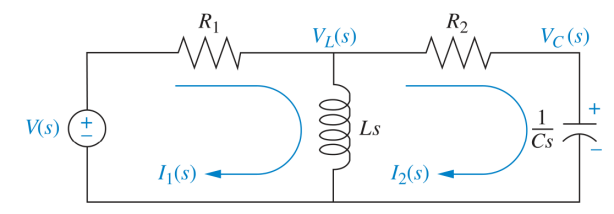


Figure 1.2: RL-RC Circuit.

consisting of a series resistor R_1 and inductor L in the first loop, and a series resistor R_2 and capacitor C in the second loop, with the inductor acting as a shared mutual impedance between the loops. We define independent loop currents $I_1(s)$ and $I_2(s)$ flowing clockwise.

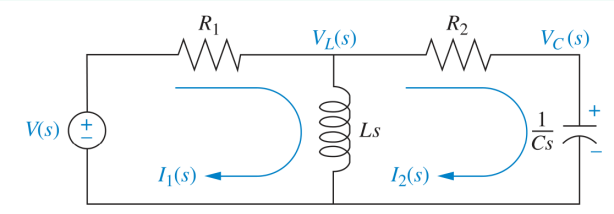


Figure 1.3: RL-RC Circuit- S - Domain.

Writing KVL around Mesh 1:

$$R_1 I_1(s) + LsI_1(s) - LsI_2(s) = V(s) \tag{1.32}$$

Writing KVL around Mesh 2:

$$LsI_2(s) + R_2 I_2(s) + \frac{1}{Cs} I_2(s) - LsI_1(s) = 0 \tag{1.33}$$

Grouping these equations yields the system:

$$(R_1 + Ls)I_1(s) - LsI_2(s) = V(s) \tag{1.34}$$

$$-LsI_1(s) + \left(Ls + R_2 + \frac{1}{Cs}\right) I_2(s) = 0 \tag{1.35}$$

Using Cramer's Rule to solve for the second loop current $I_2(s)$:

$$I_2(s) = \frac{\begin{vmatrix} R_1 + Ls & V(s) \\ -Ls & 0 \end{vmatrix}}{\Delta} = \frac{LsV(s)}{\Delta} \tag{1.36}$$

where the system determinant Δ is defined as:

$$\Delta = \begin{vmatrix} R_1 + Ls & -Ls \\ -Ls & Ls + R_2 + \frac{1}{Cs} \end{vmatrix} = (R_1 + Ls)\left(Ls + R_2 + \frac{1}{Cs}\right) - L^2 s^2 \tag{1.37}$$

Expanding and simplifying the determinant Δ:

$$\Delta = (R_1 + R_2)Ls + R_1 R_2 + \frac{R_1}{Cs} + \frac{L}{C} \tag{1.38}$$

To clear the fraction in the transfer function $G(s) = I_2(s)/V(s)$, we multiply the numerator and denominator by Cs:

$$G(s) = \frac{I_2(s)}{V(s)} = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1 R_2 C + L)s + R_1} \tag{1.39}$$

Example 1.3.3 (Two RC Network)

Find

$$\frac{E_o(s)}{E_i(s)}$$

Note:-

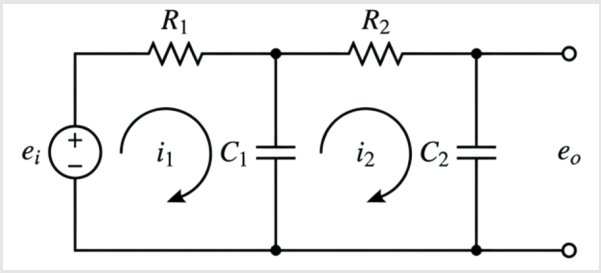


Figure 1.4: RL-RC Circuit.

1. Convert to Impedance

Redraw the circuit with all components represented by their s-domain impedances:

* Resistors: R_1, R_2 * Capacitors: $\frac{1}{C_1 s}, \frac{1}{C_2 s}$

$$\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{around Mesh 1}\end{bmatrix} I_1(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{common to the}\\ \text{two meshes}\end{bmatrix} I_2(s) = \begin{bmatrix}\text{Sum of applied}\\ \text{voltages around}\\ \text{Mesh 1}\end{bmatrix} \quad (1.40)$$

$$-\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{common to the}\\ \text{two meshes}\end{bmatrix} I_1(s) + \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{around Mesh 2}\end{bmatrix} I_2(s) = \begin{bmatrix}\text{Sum of applied}\\ \text{voltages around}\\ \text{Mesh 2}\end{bmatrix} \quad (1.41)$$

2. Write Simultaneous Equations (Mesh Analysis)
Apply Kirchhoff's Voltage Law (KVL) to each mesh to generate a system of algebraic equations in terms of $I_1(s)$ and $I_2(s)$:
Mesh 1:

$$\left(R_1 + \frac{1}{C_1 s}\right) I_1(s) - \frac{1}{C_1 s} I_2(s) = E_i(s)$$

Mesh 2:

$$-\frac{1}{C_1 s} I_1(s) + \left(R_2 + \frac{1}{C_1 s} + \frac{1}{C_2 s}\right) I_2(s) = 0$$

Output Equation:

$$E_o(s) = I_2(s) \cdot \frac{1}{C_2 s}$$

To solve the system of simultaneous equations for $I_1(s)$ and $I_2(s)$ using Cramer's Rule.
Represent the system in matrix form:

$$\begin{bmatrix} R_1 + \frac{1}{C_1 s} & -\frac{1}{C_1 s} \\ -\frac{1}{C_1 s} & R_2 + \frac{1}{C_1 s} + \frac{1}{C_2 s} \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} E_i(s) \\ 0 \end{bmatrix}$$

Step 1: Find the Determinant of the Coefficient Matrix (Δ)

$$\Delta = \begin{vmatrix} R_1 + \frac{1}{C_1 s} & -\frac{1}{C_1 s} \\ -\frac{1}{C_1 s} & R_2 + \frac{1}{C_1 s} + \frac{1}{C_2 s} \end{vmatrix}$$

Expanding the determinant:

$$\Delta = \left(R_1 + \frac{1}{C_1 s}\right)\left(R_2 + \frac{1}{C_1 s} + \frac{1}{C_2 s}\right) - \left(-\frac{1}{C_1 s}\right)\left(-\frac{1}{C_1 s}\right)$$

$$\Delta = \left(R_1 + \frac{1}{C_1 s}\right)\left(R_2 + \frac{1}{C_1 s} + \frac{1}{C_2 s}\right) - \frac{1}{C_1^2 s^2}$$

Step 2: Find the Determinant for $I_2(s)$ (Δ_2)

Using Cramer's Rule, to solve for $I_2(s)$, replace the second column of the coefficient matrix with the column vector of constants:

$$\Delta_2 = \begin{vmatrix} R_1 + \frac{1}{C_1 s} & E_i(s) \\ -\frac{1}{C_1 s} & 0 \end{vmatrix}$$

Expanding Δ_2:

$$\Delta_2 = (R_1 + \frac{1}{C_1 s})(0) - (E_i(s))\left(-\frac{1}{C_1 s}\right) = \frac{E_i(s)}{C_1 s}$$

Step 3: Solve for $I_2(s)$

$$I_2(s) = \frac{\Delta_2}{\Delta} = \frac{\frac{E_i(s)}{C_1 s}}{\left(R_1 + \frac{1}{C_1 s}\right)\left(R_2 + \frac{1}{C_1 s} + \frac{1}{C_2 s}\right) - \frac{1}{C_1^2 s^2}}$$

Multiply the numerator and denominator by $(C_1 s)(C_2 s)$ to simplify:

$$I_2(s) = \frac{C_2 s\, E_i(s)}{(R_1 C_1 s + 1)(R_2 C_2 s + 1 + \frac{C_2}{C_1}) + \ldots}$$

More cleanly, by clearing fractions across the main determinant:

$$I_2(s) = \frac{E_i(s) \cdot C_2 s}{(R_1 C_1 s + 1)(R_2 C_2 s + 1) + R_1 C_2 s}$$

Step 4: Determine the Final Transfer Function $\frac{E_o(s)}{E_i(s)}$

Given the output equation $E_o(s) = I_2(s) \cdot \frac{1}{C_2 s}$, substitute our expression for $I_2(s)$:

$$E_o(s) = \left(\frac{E_i(s) \cdot C_2 s}{(R_1 C_1 s + 1)(R_2 C_2 s + 1) + R_1 C_2 s}\right) \cdot \frac{1}{C_2 s}$$

Canceling $C_2 s$ from the numerator and denominator:

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{(R_1 C_1 s + 1)(R_2 C_2 s + 1) + R_1 C_2 s}$$

Example 1.3.4 (Three Loop System)

Three-Loop Electrical System by Inspection Application

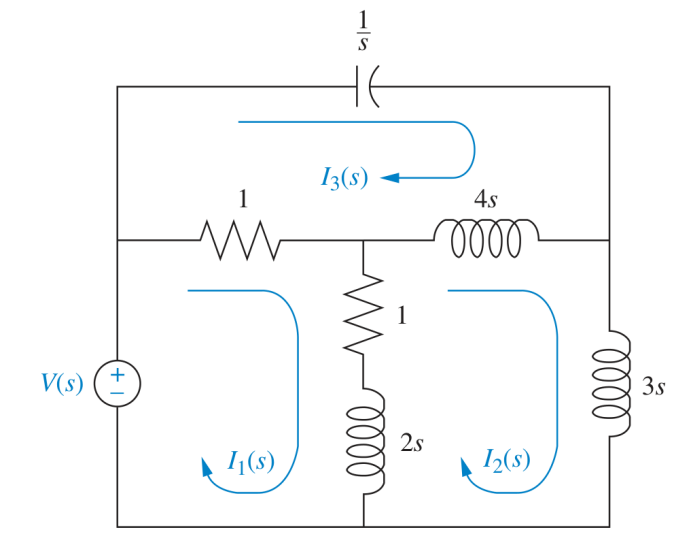


Figure 1.5: Three Loop System

To demonstrate the inspection method, we write (but do not solve) the equations of motion for the three-loop electrical network shown in Figure 1.5. Redraw the network with Laplace impedances and assume clockwise loop currents $I_1(s)$, $I_2(s)$, and $I_3(s)$:

$$\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{around Mesh 1}\end{bmatrix} I_1(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{common to meshes 1 \& 2}\end{bmatrix} I_2(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{common to meshes 1 \& 3}\end{bmatrix} I_3(\tag{1.42}$$

$$- \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{common to meshes 1 \& 2}\end{bmatrix} I_1(s) + \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{around Mesh 2}\end{bmatrix} I_2(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{common to meshes 2 \& 3}\end{bmatrix} I_3(\tag{1.43}$$

$$- \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{common to meshes 1 \& 3}\end{bmatrix} I_1(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{common to meshes 2 \& 3}\end{bmatrix} I_2(s) + \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{around Mesh 3}\end{bmatrix} I_3(\tag{1.44}$$

Applying the inspection template for Mesh 1, 2, and 3:

$$(2s+2)\,I_1(s) - (2s+1)\,I_2(s) - I_3(s) = V(s) \tag{1.45}$$

$$-(2s+1)\,I_1(s) + (9s+1)\,I_2(s) - 4sI_3(s) = 0 \tag{1.46}$$

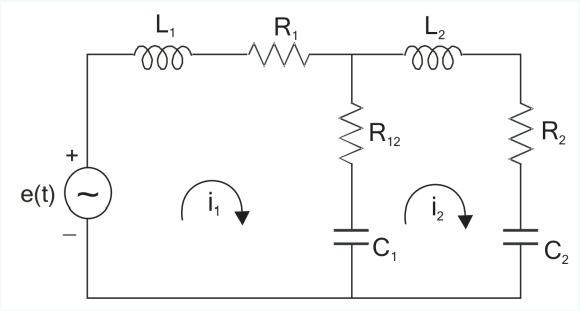
$$-I_1(s) - 4sI_2(s) + \left(4s + 1 + \frac{1}{s}\right) I_3(s) = 0 \tag{1.47}$$

Note:-

Complete the Remaining Calculation. Hint: Apply Cramer's Rule

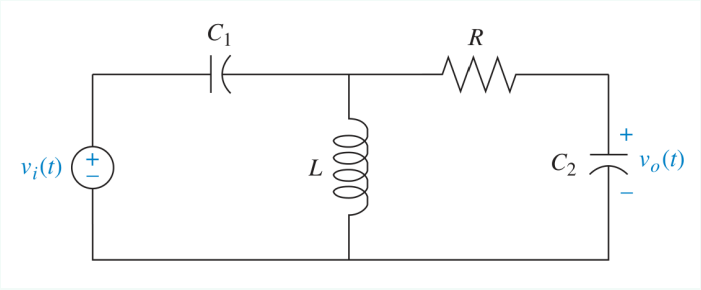
Example 1.3.5 (Exercise Problem 1)

Find the transfer function, $G(s) = I_2(s)/V(s)$, for each network shown in Figure



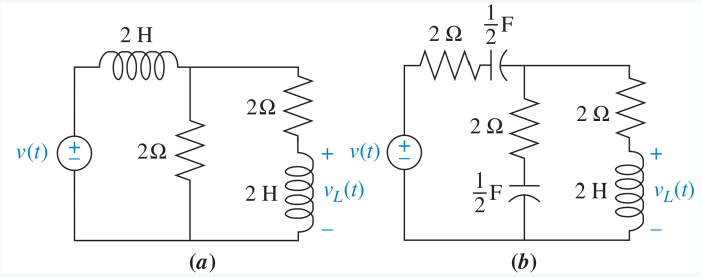
Example 1.3.6 (Exercise Problem 2)

Find the transfer function, $G(s) = V_o(s)/V(s)$, for each network shown in Figure



Example 1.3.7 (Exercise Problem 3)

Find the transfer function, $G(s) = V_L(s)/V(s)$, for each network shown in Figure


2 H
2Ω
2Ω
2 H
v(t)
+
v_L(t)
−
(a)
2 Ω
1/2 F
2 Ω
2 Ω
1/2 F
2 H
v(t)
+
v_L(t)
−
(b)

1.3.2 Modeling of Mechanical System

Example 1.3.8 (Two Mass System)

Problem: Consider the translational mechanical system shown below. Draw the Free Body Diagrams (FBD) for each node and derive the transfer function $\frac{X(s)}{F(s)}$.

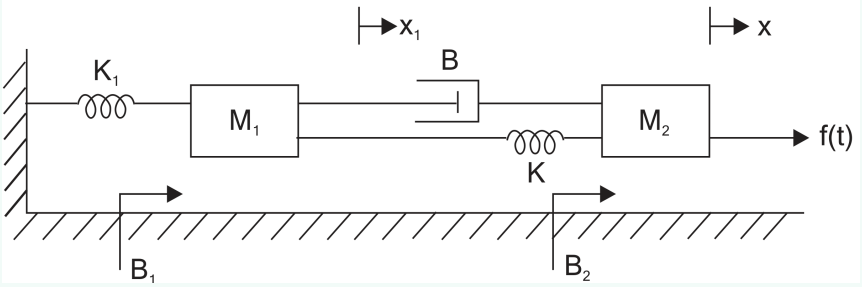
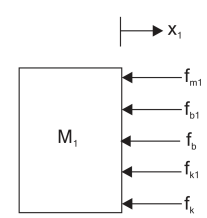


Figure 1.6: Translational mechanical system with two masses.

Solution:

The diagrams use D'Alembert's principle, where the inertial forces (f_{m1} and f_{m2}) are drawn as forces opposing the direction of motion.

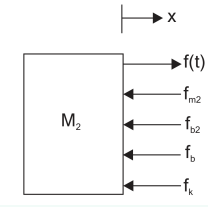
Free Body Diagram of Node 1 (M_1)



For the first mass, all labeled forces act to the left, opposing the positive displacement direction x_1. Setting the sum of the forces to zero gives:

$$f_{m1} + f_{b1} + f_b + f_{k1} + f_k = 0$$

Free Body Diagram of Node 2 (M_2)



For the second mass, the applied force $f(t)$ acts to the right in the direction of positive displacement x, while the remaining forces act to the left. Balancing the rightward and leftward forces gives:

$$f_{m2} + f_{b2} + f_b + f_k = f(t)$$

First, let's define the individual forces based on the displacement x_1 of mass M_1 and the displacement x of mass M_2:

- Inertial forces (Mass): Proportional to acceleration.
  - $f_{m1} = M_1 \frac{d^2x_1}{dt^2}$
  - $f_{m2} = M_2 \frac{d^2x}{dt^2}$
- Damping forces (Dashpot/Friction): Proportional to velocity.
  - $f_{b1} = B_1 \frac{dx_1}{dt}$ (damping on M_1 relative to a fixed reference)
  - $f_{b2} = B_2 \frac{dx}{dt}$ (damping on M_2 relative to a fixed reference)
  - f_b is the relative damping between the two masses. For M_1, it is $B\left(\frac{dx_1}{dt} - \frac{dx}{dt}\right)$. For M_2, it is $B\left(\frac{dx}{dt} - \frac{dx_1}{dt}\right)$.
- Spring forces (Stiffness): Proportional to displacement.
  - $f_{k1} = K_1 x_1$ (spring on M_1 relative to a fixed reference)
  - f_k is the relative spring force between the two masses. For M_1, it is $K(x_1 - x)$. For M_2, it is $K(x - x_1)$.

Substituting these into your previous force balance equations yields the final ODEs:
Differential Equation for Node 1 (M_1)

$$M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B\left(\frac{dx_1}{dt} - \frac{dx}{dt}\right) + K_1 x_1 + K(x_1 - x) = 0$$

Differential Equation for Node 2 (M_2)

$$M_2 \frac{d^2x}{dt^2} + B_2 \frac{dx}{dt} + B\left(\frac{dx}{dt} - \frac{dx_1}{dt}\right) + K(x - x_1) = f(t)$$

1. Equations of Motion in the s-Domain

Taking the Laplace transform of the time-domain differential equations (assuming zero initial conditions):

$$\left[M_1 s^2 + (B + B_1)s + (K_1 + K)\right] X_1(s) - (Bs + K)\, X(s) = 0$$
$$\left[M_2 s^2 + (B + B_2)s + K\right] X(s) - (Bs + K)\, X_1(s) = F(s)$$

2. Solving for $X_1(s)$ from Equation 1

From the first equation, move the coupling terms to the right-hand side:

$$\left[M_1 s^2 + (B + B_1)s + (K_1 + K)\right] X_1(s) = (Bs + K)X(s)$$

Divide both sides by the coefficient of $X_1(s)$ to isolate it:

$$X_1(s) = \frac{Bs + K}{M_1 s^2 + (B + B_1)s + (K_1 + K)} X(s)$$

3. Substituting $X_1(s)$ into Equation 2

Substitute the isolated expression into the second equation:

$$\left[M_2 s^2 + (B + B_2)s + K\right] X(s) - (Bs + K)\left[\frac{Bs + K}{M_1 s^2 + (B + B_1)s + (K_1 + K)} X(s)\right] = F(s)$$

Combine the numerator terms $(Bs + K)$:

$$\left[M_2 s^2 + (B + B_2)s + K\right] X(s) - \frac{(Bs + K)^2}{M_1 s^2 + (B + B_1)s + (K_1 + K)} X(s) = F(s)$$

4. Factoring out $X(s)$

Factor $X(s)$ out from the left-hand side:

$$X(s)\left[\left(M_2 s^2 + (B + B_2)s + K\right) - \frac{(Bs + K)^2}{M_1 s^2 + (B + B_1)s + (K_1 + K)}\right] = F(s)$$

5. Finding a Common Denominator

To subtract the terms inside the brackets, find a common denominator:

$$X(s)\left[\frac{\left(M_1 s^2 + (B + B_1)s + (K_1 + K)\right)\left(M_2 s^2 + (B + B_2)s + K\right) - (Bs + K)^2}{M_1 s^2 + (B + B_1)s + (K_1 + K)}\right] = F(s)$$

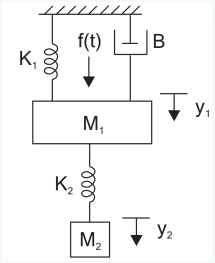
6. Final Transfer Function $\frac{X(s)}{F(s)}$

Rearrange the equation to solve for the ratio:

$$\frac{X(s)}{F(s)} = \frac{M_1 s^2 + (B + B_1)s + (K_1 + K)}{\left[M_1 s^2 + (B + B_1)s + (K_1 + K)\right]\left[M_2 s^2 + (B + B_2)s + K\right] - (Bs + K)^2}$$

Example 1.3.9 (Mechanical System Modeling)

Problem: Consider the translational mechanical system shown below. Draw the Free Body Diagrams (FBD) for each node and derive the transfer function $\frac{Y_2(s)}{F(s)}$.



Solution:

The diagrams use D'Alembert's principle, where the inertial forces are drawn opposing the positive direction of motion (downward).

Free Body Diagram of Node 1 (M_1)

For the first mass, the applied force $f(t)$ acts downward in the positive y_1 direction. The opposing forces act upward. Setting the sum of the forces to zero gives:

$$f_{m1} + f_b + f_{k1} + f_{k2} = f(t)$$

Free Body Diagram of Node 2 (M_2)

For the second mass, there is no external applied force. The forces act to oppose the positive y_2 direction. Balancing the forces gives:

$$f_{m2} + f'_{k2} = 0$$

First, let's define the individual forces based on the displacement y_1 of mass M_1 and y_2 of mass M_2:

- Inertial forces (Mass): Proportional to acceleration.
  - $f_{m1} = M_1 \frac{d^2 y_1}{dt^2}$
  - $f_{m2} = M_2 \frac{d^2 y_2}{dt^2}$
- Damping forces (Dashpot/Friction): Proportional to velocity.
  - $f_b = B \frac{dy_1}{dt}$ (damping on M_1 relative to a fixed reference)
- Spring forces (Stiffness): Proportional to displacement.

- $f_{k1} = K_1 y_1$ (spring on M_1 relative to a fixed reference)
- f_{k2} is the relative spring force between the two masses. For M_1, it is $K_2(y_1 - y_2)$. For M_2, the opposing internal force is $K_2(y_2 - y_1)$.

Substituting these into the force balance equations yields the final ODEs:
Differential Equation for Node 1 (M_1)

$$M_1 \frac{d^2 y_1}{dt^2} + B\frac{dy_1}{dt} + K_1 y_1 + K_2(y_1 - y_2) = f(t)$$

Differential Equation for Node 2 (M_2)

$$M_2 \frac{d^2 y_2}{dt^2} + K_2(y_2 - y_1) = 0$$

1. Equations of Motion in the s-Domain

Taking the Laplace transform of the time-domain differential equations (assuming zero initial conditions):

$$\begin{aligned} \left[M_1 s^2 + Bs + (K_1 + K_2)\right] Y_1(s) - K_2 Y_2(s) &= F(s) \\ \left[M_2 s^2 + K_2\right] Y_2(s) - K_2 Y_1(s) &= 0 \end{aligned}$$

2. Solving for $Y_1(s)$ from Equation 2

From the second equation, move the coupling terms to the right-hand side:

$$\left(M_2 s^2 + K_2\right) Y_2(s) = K_2 Y_1(s)$$

Divide both sides by the coefficient of $Y_1(s)$ to isolate it:

$$Y_1(s) = \frac{M_2 s^2 + K_2}{K_2} Y_2(s)$$

3. Substituting $Y_1(s)$ into Equation 1

Substitute the isolated expression into the first equation:

$$\left[M_1 s^2 + Bs + (K_1 + K_2)\right] \left(\frac{M_2 s^2 + K_2}{K_2} Y_2(s)\right) - K_2 Y_2(s) = F(s)$$

4. Factoring out $Y_2(s)$

Factor $Y_2(s)$ out from the left-hand side:

$$Y_2(s) \left[\frac{\left(M_1 s^2 + Bs + (K_1 + K_2)\right)\left(M_2 s^2 + K_2\right)}{K_2} - K_2\right] = F(s)$$

5. Finding a Common Denominator

To subtract the terms inside the brackets, find a common denominator:

$$Y_2(s)\left[\frac{\left(M_1s^2+Bs+(K_1+K_2)\right)\left(M_2s^2+K_2\right)-K_2^2}{K_2}\right]=F(s)$$

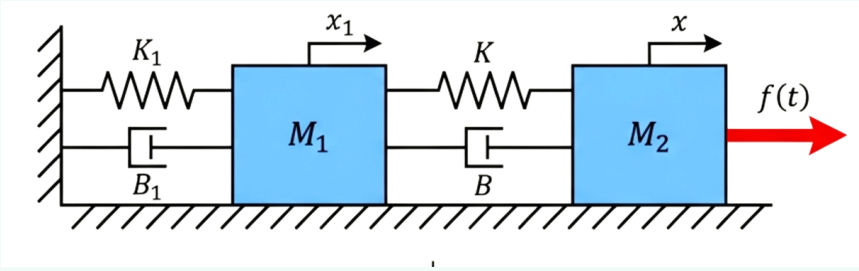
6. Final Transfer Function $\frac{Y_2(s)}{F(s)}$

Rearrange the equation to solve for the ratio:

$$\frac{Y_2(s)}{F(s)}=\frac{K_2}{\left[M_1s^2+Bs+(K_1+K_2)\right]\left(M_2s^2+K_2\right)-K_2^2}$$

Example 1.3.10 (Example of Mechanical System modeling)

Problem: Consider the translational mechanical system shown below. Draw the Free Body Diagrams (FBD) for each node and derive the transfer function $\frac{X_2(s)}{F(s)}$.



Solution:

The diagrams use D'Alembert's principle, where the inertial forces are drawn opposing the direction of positive motion.

Free Body Diagram of Node 1 (M_1)

For the first mass, all internal forces act to the left, opposing the positive displacement x_1. Setting the sum of the forces to zero gives:

$$f_{m1}+f_{b1}+f_b+f_{k1}+f_k=0$$

Free Body Diagram of Node 2 (M_2)

For the second mass, the applied force $f(t)$ acts to the right, while the other components react to the left. Balancing the forces gives:

$$f_{m2}+f_{b2}+f_b'+f_k'=f(t)$$

First, define the individual forces based on the displacement x_1 of mass M_1 and x_2 of mass M_2:

- Inertial forces:
  - $f_{m1} = M_1 \frac{d^2x_1}{dt^2}$
  - $f_{m2} = M_2 \frac{d^2x_2}{dt^2}$
- Damping forces:
  - $f_{b1} = B_1 \frac{dx_1}{dt}$
  - $f_{b2} = B_2 \frac{dx_2}{dt}$
  - f_b and f_b' represent relative damping: $B\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right)$ for M_1, and $B\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right)$ for M_2.
- Spring forces:
  - $f_{k1} = K_1 x_1$
  - f_k and f_k' represent relative spring forces: $K(x_1 - x_2)$ for M_1, and $K(x_2 - x_1)$ for M_2.

Substituting these yields the final ODEs:
Differential Equation for Node 1 (M_1)

$$M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right) + K_1 x_1 + K(x_1 - x_2) = 0$$

Differential Equation for Node 2 (M_2)

$$M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + B\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right) + K(x_2 - x_1) = f(t)$$

1. Equations of Motion in the s-Domain

Taking the Laplace transform (assuming zero initial conditions):

$$\left[M_1 s^2 + (B_1 + B)s + (K_1 + K)\right] X_1(s) - (Bs + K) X_2(s) = 0$$
$$\left[M_2 s^2 + (B_2 + B)s + K\right] X_2(s) - (Bs + K) X_1(s) = F(s)$$

2. Solving for $X_1(s)$ from Equation 1

Move the coupling terms to the right-hand side:

$$\left[M_1 s^2 + (B_1 + B)s + (K_1 + K)\right] X_1(s) = (Bs + K)X_2(s)$$

Divide to isolate $X_1(s)$:

$$X_1(s) = \frac{Bs + K}{M_1 s^2 + (B_1 + B)s + (K_1 + K)} X_2(s)$$

3. Substituting $X_1(s)$ into Equation 2

Substitute this expression into the second equation:

$$\left[M_2 s^2 + (B_2 + B)s + K\right] X_2(s) - (Bs + K)\left[\frac{Bs + K}{M_1 s^2 + (B_1 + B)s + (K_1 + K)} X_2(s)\right] = F(s)$$

4. Factoring out $X_2(s)$

Factor $X_2(s)$ out from the left-hand side:

$$X_2(s)\left[\left(M_2 s^2 + (B_2 + B)s + K\right) - \frac{(Bs + K)^2}{M_1 s^2 + (B_1 + B)s + (K_1 + K)}\right] = F(s)$$

5. Finding a Common Denominator

Find a common denominator for the bracketed term:

$$X_2(s)\left[\frac{\left(M_1 s^2 + (B_1 + B)s + (K_1 + K)\right)\left(M_2 s^2 + (B_2 + B)s + K\right) - (Bs + K)^2}{M_1 s^2 + (B_1 + B)s + (K_1 + K)}\right] = F(s)$$

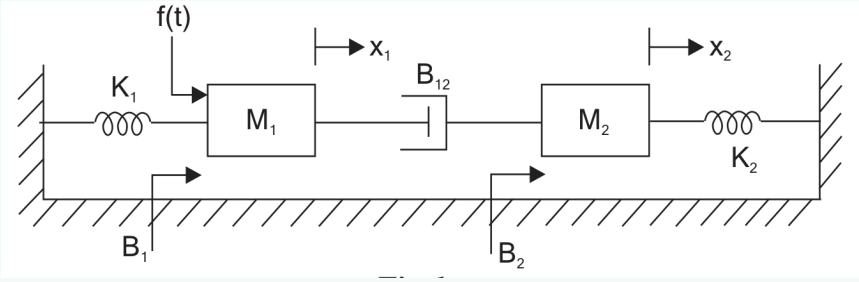
6. Final Transfer Function $\frac{X_2(s)}{F(s)}$

Rearrange to solve for the ratio:

$$\frac{X_2(s)}{F(s)} = \frac{M_1 s^2 + (B_1 + B)s + (K_1 + K)}{\left[M_1 s^2 + (B_1 + B)s + (K_1 + K)\right]\left[M_2 s^2 + (B_2 + B)s + K\right] - (Bs + K)^2}$$

Example 1.3.11 (ex4)

Problem: Consider the translational mechanical system shown below. Draw the Free Body Diagrams (FBD) for each node and derive the transfer functions $\frac{X_1(s)}{F(s)}$ and $\frac{X_2(s)}{F(s)}$.



Solution:

The diagrams use D'Alembert's principle, placing inertial forces opposite to motion.

Free Body Diagram of Node 1 (M_1)

For the first mass, the external force acts to the right, and opposing forces react to the left. Setting the sum to zero gives:

$$f_{m1} + f_{b1} + f_{b12} + f_{k1} = f(t)$$

Free Body Diagram of Node 2 (M_2)

For the second mass, opposing forces react to the relative motion:

$$f_{m2} + f_{b2} + f'_{b12} + f_{k2} = 0$$

Define the individual forces based on the displacements x_1 and x_2:

- Inertial forces: $f_{m1} = M_1 \frac{d^2x_1}{dt^2},\ f_{m2} = M_2 \frac{d^2x_2}{dt^2}$
- Damping forces: $f_{b1} = B_1 \frac{dx_1}{dt},\ f_{b2} = B_2 \frac{dx_2}{dt}$
- Relative damping: $f_{b12} = B_{12}\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right),\ f'_{b12} = B_{12}\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right)$
- Spring forces: $f_{k1} = K_1x_1,\ f_{k2} = K_2x_2$

Substituting these yields the ODEs:

Differential Equation for Node 1 (M_1)

$$M_1\frac{d^2x_1}{dt^2} + B_1\frac{dx_1}{dt} + B_{12}\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right) + K_1x_1 = f(t)$$

Differential Equation for Node 2 (M_2)

$$M_2\frac{d^2x_2}{dt^2} + B_2\frac{dx_2}{dt} + B_{12}\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right) + K_2x_2 = 0$$

1. Equations of Motion in the s-Domain

Taking the Laplace transform:

$$\left[M_1 s^2 + (B_1 + B_{12})s + K_1\right] X_1(s) - (B_{12}s)X_2(s) = F(s)$$
$$\left[M_2 s^2 + (B_2 + B_{12})s + K_2\right] X_2(s) - (B_{12}s)X_1(s) = 0$$

2. Solving for $X_2(s)$ from Equation 2

Move the coupling terms to the right-hand side:

$$\left[M_2 s^2 + (B_2 + B_{12})s + K_2\right] X_2(s) = (B_{12}s)X_1(s)$$

Divide to isolate $X_2(s)$:

$$X_2(s) = \frac{B_{12}s}{M_2 s^2 + (B_2 + B_{12})s + K_2} X_1(s)$$

3. Substituting $X_2(s)$ into Equation 1

Substitute this expression into the first equation:

$$\left[M_1 s^2 + (B_1 + B_{12})s + K_1\right] X_1(s) - (B_{12}s)\left[\frac{B_{12}s}{M_2 s^2 + (B_2 + B_{12})s + K_2} X_1(s)\right] = F(s)$$

4. Factoring out $X_1(s)$

Factor $X_1(s)$ out from the left-hand side:

$$X_1(s)\left[\left(M_1 s^2 + (B_1 + B_{12})s + K_1\right) - \frac{(B_{12}s)^2}{M_2 s^2 + (B_2 + B_{12})s + K_2}\right] = F(s)$$

5. Finding a Common Denominator

Find a common denominator:

$$X_1(s)\left[\frac{\left(M_1 s^2 + (B_1 + B_{12})s + K_1\right)\left(M_2 s^2 + (B_2 + B_{12})s + K_2\right) - (B_{12}s)^2}{M_2 s^2 + (B_2 + B_{12})s + K_2}\right] = F(s)$$

6. Final Transfer Function $\frac{X_1(s)}{F(s)}$

Rearrange to solve for the first ratio:

$$\frac{X_1(s)}{F(s)} = \frac{M_2 s^2 + (B_2 + B_{12})s + K_2}{\left[M_1 s^2 + (B_1 + B_{12})s + K_1\right]\left[M_2 s^2 + (B_2 + B_{12})s + K_2\right] - (B_{12}s)^2}$$

7. Solving for $X_1(s)$ from Equation 2

To find the second transfer function, return to the s-Domain equations and solve Equation 2 for $X_1(s)$:

$$X_1(s) = \frac{M_2 s^2 + (B_2 + B_{12})s + K_2}{B_{12}s} X_2(s)$$

8. Substituting $X_1(s)$ into Equation 1

$$\left[M_1 s^2 + (B_1 + B_{12})s + K_1\right] \left(\frac{M_2 s^2 + (B_2 + B_{12})s + K_2}{B_{12}s} X_2(s)\right) - (B_{12}s)X_2(s) = F(s)$$

9. Factoring out $X_2(s)$

$$X_2(s) \left[\frac{\left(M_1 s^2 + (B_1 + B_{12})s + K_1\right)\left(M_2 s^2 + (B_2 + B_{12})s + K_2\right)}{B_{12}s} - (B_{12}s)\right] = F(s)$$

10. Finding a Common Denominator

$$X_2(s) \left[\frac{\left(M_1 s^2 + (B_1 + B_{12})s + K_1\right)\left(M_2 s^2 + (B_2 + B_{12})s + K_2\right) - (B_{12}s)^2}{B_{12}s}\right] = F(s)$$

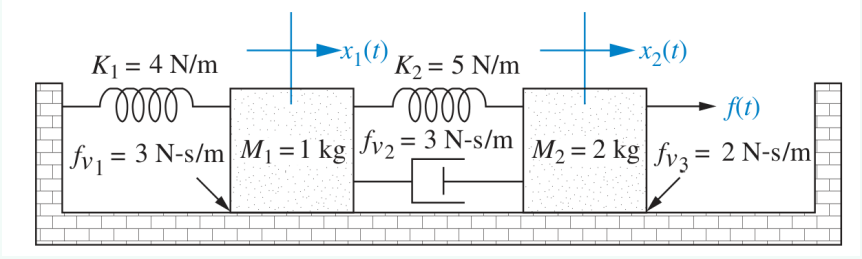
11. Final Transfer Function $\frac{X_2(s)}{F(s)}$

Rearrange to solve for the second ratio:

$$\frac{X_2(s)}{F(s)} = \frac{B_{12}s}{\left[M_1 s^2 + (B_1 + B_{12})s + K_1\right]\left[M_2 s^2 + (B_2 + B_{12})s + K_2\right] - (B_{12}s)^2}$$

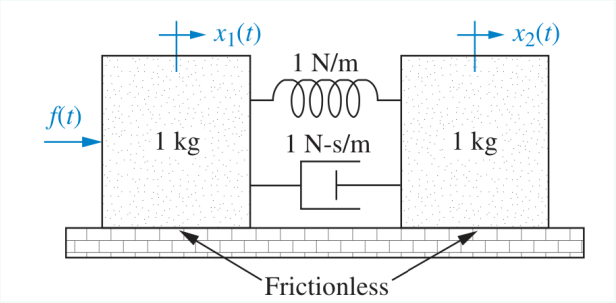
Example 1.3.12 (Exercise Problem 1)

Find the transfer function, $G(s) = X_1(s)/F(s)$, for each network shown in Figure



Example 1.3.13 (Exercise Problem 2)

Find the transfer function, $G(s) = V_o(s)/V(s)$, for each network shown in Figure


$x_1(t)$
$x_2(t)$
1 N/m
$f(t)$
1 kg
1 N-s/m
1 kg
Frictionless